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ON l_∞-ESCAPE IN A LINEAR MANY-PERSON DIFFERENTIAL GAME WITH INTEGRAL CONSTRAINTS*

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A criterion of guaranteed evasion from contact with an estimate not tending to zero as $t \to +\infty$ is derived for a linear many-person differential game with integral constraints. The possibility is proved of l-escape and of l_∞-escape in a broad class of linear differential games. The paper borders on the studies in /1—8/.

1. Let the motion of a vector z in an n-dimensional Euclidean space R be governed by the vector differential equation

$$dz / dt = Cz - u + v, \quad u \in P, \quad v \in Q \tag{1.1}$$

Here C is a constant nth-order square matrix, $u = u(t)$ is the control vector of pursuer U, $v = v(t)$ is the control vector of evader V, P and Q are linear subspaces in R, $\dim Q \geqslant 3$. Finite measurable vector-valued functions $v(t)$ and $u(t)$ square summable in modulus on each finite interval and satisfying the relations

$$\int_0^t |u(s)|^2 ds \leqslant \rho^2, \quad \int_0^t |v(s)|^2 ds \leqslant \sigma^2, \quad u(t) \in P, \quad v(t) \in Q, \quad \rho = \text{const} > 0, \quad \sigma = \text{const} > 0 \tag{1.2}$$

for each $t \geqslant 0$ are everywhere called the controls of the players (of the evader and the pursuer, respectively). Suppose that a terminal set M has been prescribed in R, being the union of linear subspaces M_i $(i = 1, \ldots, m)$. The pursuer's purpose is to take point $z(t)$ onto the terminal set M; the evader's purpose is to guarantee evasion from set M. We say /1/ that the many-person differential game (1.1) with integral constraints has been defined by all the data listed.

Definition. We say that l_∞-escape is possible in game (1.1) if for any initial state $z_0 = z(0) \in R \setminus M$ the escaper, knowing all the data describing game (1.1) and the values of $z(s)$, $s \in [0, t]$ and of $u(t)$ at each instant t, can construct, for any pursuer's control $u^* = \{u(t),\ t \geqslant 0\}$ his own control $v^* = \{v(t),\ t \geqslant 0\}$ so as to ensure the estimate

$$\rho(z(t)) \geqslant l(t) \tag{1.3}$$

for $t \geqslant 0$, where $l(t) > 0$ $(t \in (0, +\infty))$ is a function dependent only on game (1.1) and independent of z_0, such that $l(t) \to \infty$ as $t \to +\infty$; $\rho(z) = \operatorname{dist}(z, M)$ is the distance from point z to set M.

An estimate of type (1.3) is called a guaranteed estimate. If for game (1.1) numbers $l > 0$ and $\theta > 0$ exist such that

$$l(t) \geqslant l, \quad t \geqslant \theta \tag{1.4}$$

in the presence of estimate (1.3), then we say that an l-escape obtains.

For the l-escape problem posed in /2/ necessary and sufficient conditions have been obtain /3,4/ for stationary differential games with geometric constraints. For many-person games with integral constraints the first guaranteed estimate was obtained in /5/ (in /5/ in the general case $l(t) \to 0$, $t \to +\infty$, but the l-escape problem still was not solved). In the present paper, for a sufficiently broad class of linear differential games, it is proved that l_∞-escape is possible under the fulfillment of Conditions 1—4 (see Sect.2 below) and, as an obvious corollary, so is l-escape (l_∞-escape corresponds to the possibility of guaranteeing the evader an evasion of the terminal set by any preassigned distance l, i.e., an l-escape in which number l now is independent of game (1.1)).

By L_i we denote the orthogonal complement to M_i in R. We assume that $\dim L_i \geqslant 3$ $(i = 1, \ldots, m)$ and we fix three-dimensional subspaces $W_i \subset L_i$ $(i = 1, \ldots, m)$ and a three-dimensional space $Q^* \subset Q$. By π_i $(i = 1, \ldots, m)$, π_* and π^* we denote orthogonal projections operators from R onto W_i, P and Q^*, respectively; by $\Phi(r)$ we denote the matrix e^{rC}; by $\|B\|$ we denote the norm of linear operator B; by K we denote the unit sphere in Q^*.

2. Let us formulate the escape conditions.

Condition 1. All eigenvalues of matrix C are real.

Condition 2. For each $i = 1, \ldots, m$ there exist nonsingular constant linear operators

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A_i acting from Q^* onto W_i and continuous functions $\delta_i(t) \geqslant 0$ nonnegative on $[0, +\infty)$ such that

$$\pi_i \Phi(t) \pi^* \equiv \delta_i(t) A_i \pi^* \tag{2.1}$$

It is well known /6/ that in this case each of the functions $\delta_i(t)$ is a quasipolynomial

$$\delta_i(t) = e^{\lambda_1^i t} P_1^i(t) + \ldots + e^{\lambda_{r_i}^i t} P_{r_i}^i(t) \quad (i = 1, \ldots, m) \tag{2.2}$$

in which $\lambda_1^i > \lambda_2^i > \ldots > \lambda_{r_i}^i$ are the eigenvalues of matrix C, while the degree of each of the polynomials $P_j^i(t)$ is less by one (or more) than the multiplicity of eigenvalue λ_j^i.

Condition 3. For each $i = 1, \ldots, m$, $\lambda_1^i \geqslant 0$ for the quasipolynomial $\delta_i(t)$ given by Condition 2.

We set

$$\gamma_i(r) = \| A_i^{-1} \pi_i \Phi(r) \pi_* \|, \ r \geqslant 0, \quad H_i(t, z) = A_i^{-1} \pi_i \Phi(t) z, \quad t \geqslant 0, \ z \in R$$

Lemma 1 /7/. If in the quasipolynomial $h(t) = e^{\lambda_1 t} P_1(t) + \ldots + e^{\lambda_k t} P_k(t)$ all λ_i $(i = 1, \ldots, k)$ are real, while the degree of each of the polynomials $P_i(t)$ equals s_i $(i = 1, \ldots, k)$, respectively, then $h(t)$ has no more that $k - 1 + s_1 + \ldots + s_k$ zeros on the real axis.

By virtue of this lemma and of Condition 1, for each $i = 1, \ldots, m$ a number $N_i \leqslant n$ exists such that the number of zeros of the scalar function $(e \cdot H_i(t, z))$ on the semiaxis $t \in [0, +\infty)$ does not exceed N_i for fixed $z \in R$ and $e \in K$ (to prove this it is enough to note that $(e \cdot H_i(t, z))$ is a quasipolynomial of type (2.2)).

Lemma 2. (cf. Lemma 1 in /1/). For any fixed $z \in R$ there exists a vector $e(z) \in K$ such that

$$| H_i(t, z) - \lambda e(z) | \geqslant | \lambda | / \Gamma(N_1, \ldots, N_m), \quad i = 1, \ldots, m, \quad t \in [0, +\infty) \tag{2.3}$$

$$\sqrt{3}\, \Gamma(N_1, \ldots, N_m) = 18 \sum_{i=1}^m N_i + 22$$

for all real λ.

Condition 4.

$$\max_{i=1,\ldots,m} \sup_{r \in [0, +\infty)} \gamma_i(r) \Gamma(N_1, \ldots, N_m) / \delta_i(r) = \mu \in \left(0, \frac{\sigma}{\rho}\right) \tag{2.4}$$

Let

$$\omega(t) = (1 + t)^{-1}, \quad t \geqslant 0, \quad L_i(t) = \int_0^t \delta_i(t - s) \omega(s)\, ds, \qquad L(t) = \min\{L_1(t), \ldots, L_m(t)\}$$

Lemma 3. If condition 3 is fulfilled, then $L(t) \to +\infty$ as $t \to +\infty$.

Proof. Obviously, it is sufficient to verify that $L_1(t) \to +\infty$ as $t \to +\infty$. Let q be the degree of polynomial $P_1^1(t)$, i.e., $P_1^1(t) = A_0^* t^q + \ldots + A_q^*$. Since $\lambda_1^1 > \lambda_2^1 > \ldots > \lambda_{r_1}^1$, $\delta_1(t) [A_0^* t^q e^{\lambda_1^1 t}]^{-1} \to 1$ as $t \to +\infty$. Consequently, $A_0^* > 0$ and $t_1 \geqslant 1$ exists such that the inequality

$$\delta_1(t) \geqslant {}^1/_2 A_0^* t^q e^{\lambda_1^1 t} \geqslant {}^1/_2 A_0^*$$

is fulfilled for all $t \geqslant t_1$ (in the last inequality we used the fact that $\lambda_1^1 \geqslant 0$). For $t \geqslant t_1$ we have

$$L_1(t) \geqslant \int_0^{t - t_1} \frac{\delta_1(t - s)}{1 + s} ds \geqslant \frac{A_0^*}{2} \int_0^{t - t_1} \frac{ds}{1 + s} \to +\infty, \ t \to +\infty$$

3. Theorem. Let Conditions 1—4 be fulfilled for game (1.1). Then l_∞-escape is possible in game (1.1).

Proof. By the Cauchy formula we have

$$\pi_i z(t) = \pi_i \Phi(t) z_0 + \int_0^t \pi_i \Phi(t - s)[v(s) - u(s)]\, ds = A_i H_i(t, z_0) + \int_0^t \delta_i(t - s) A_i v(s)\, ds - \int_0^t \pi_i \Phi(t - s) u(s)\, ds$$

Applying the operator A_i^{-1} to both sides of this equality, we obtain

$$| A_i^{-1} \pi_i z(t) | \geqslant | H_i(t, z_0) + \int_0^t \delta_i(t - s) v(s)\, ds | - \int_0^t \gamma_i(t - s) | u(s) |\, ds \tag{3.1}$$

Let the constant $\Delta > 0$ be such that

$$\mu = \sigma / (\rho + \Delta) \tag{3.2}$$

By Lemma 2 a vector $e_0 = e(z_0)$ exists such that estimate (2.3) is valid for all $i = 1, \ldots, m$ (when $z = z_0$). We set

$$v(s) = -\mu e_0 (| u(s) | + \Delta \omega(s)), \quad s \geqslant 0$$

Then for any control u^* of player U

$$\int_0^t |v(s)|^2 ds \leqslant \mu^2 \Big[\Big(\int_0^t |u(s)|^2 ds\Big)^{1/2} + \Delta\Big(\int_0^t \frac{ds}{(1+s)^2}\Big)^{1/2}\Big]^2 \leqslant \mu^2(\rho+\Delta)^2 = \sigma^2 \tag{3.3}$$

so that the integral constraint is fulfilled. In addition, by virtue of (2.3) and (2.4)

$$\Big|H_i(t, z_0) + \int_0^t \delta_i(t-s)v(s)\,ds\Big| \geqslant \frac{\mu}{\Gamma(N_1,\ldots,N_m)}\Big\{\int_0^t \delta_i(t-s)|u(s)|\,ds + \Delta L_i(t)\Big\} \geqslant \int_0^t \gamma_i(t-s)|u(s)|\,ds + \frac{\mu\Delta}{\Gamma(N_1,\ldots,N_m)} L_i(t)$$

Therefore, inequality (3.1) yields the estimate

$$|\pi_i z(t)| \geqslant |A_i^{-1}\pi_i z(t)|\,\|A_i^{-1}\|^{-1} \geqslant l(t),\ \ t \geqslant 0,\ \ i = 1,\ldots,m, \quad l(t) = \frac{\mu\Delta L(t)}{\Gamma(N_1,\ldots,N_m)\max\limits_{i=1,\ldots,m}\|A_i^{-1}\|} \to +\infty,\ t\to+\infty$$

by virtue of Lemma 3. The theorem is proved.

4. As an example we consider the escape problem for one pursued object y whose motion is specified by the linear vector differential equation

$$y^{(k)} + a_1 y^{(k-1)} + \ldots + a_k y = v \tag{4.1}$$

from m pursuing objects x_i $(i = 1,\ldots,m)$ each of whose motions is specified by the equation

$$x_i^{(k_i)} + b_1^i x_i^{(k_i-1)} + \ldots + b_{k_i}^i x_i = u_i \tag{4.2}$$

In formulas (4.1) and (4.2) a_j $(j = 1,\ldots,k)$ and b_j^i $(j = 1,\ldots,k_i,\ i = 1,\ldots,m)$ are real numbers, v, u_i, y, x_i are ν-dimensional vectors in a Euclidean space E $(\dim E = \nu \geqslant 3)$, u_i and v are control parameters subject to the constraints (for each $t \geqslant 0$)

$$\int_0^t |v(s)|^2 ds \leqslant \sigma^2;\quad \int_0^t |u_i(s)|^2 ds \leqslant \rho_i^2,\quad (\sigma = \text{const} > 0,\ \ \rho_i = \text{const} > 0\ \ (i = 1,\ldots,m)) \tag{4.3}$$

The pursuit is considered ended when

$$\rho(t) = \min_{i=1,\ldots,m} |x_i(t) - y(t)| = 0$$

Let us verify Conditions 1—4 for problem (4.1)—(4.3). By $\delta(t)$ we denote a solution of the homogeneous scalar differential equation

$$\delta^{(k)} + a_1\delta^{(k-1)} + \ldots + a_k\delta = 0 \tag{4.4}$$

with initial condition

$$\delta(0) = \delta'(0) = \ldots = \delta^{(k-2)}(0) = 0,\quad \delta^{(k-1)}(0) = 1 \tag{4.5}$$

By $\gamma_i(t)$ we denote a solution of the equation

$$\gamma^{(k_i)} + b_1^i\gamma^{(k_i-1)} + \ldots + b_{k_i}^i\gamma = 0 \tag{4.6}$$

$$\gamma_i(0) = \gamma_i'(0) = \ldots = \gamma_i^{(k_i-2)}(0) = 0,\quad \gamma_i^{(k_i-1)}(0) = 1 \tag{4.7}$$

Reducing problem (4.1)—(4.3) in standard fashion (see /8/) to the linear game (1.1), we get that Condition 1 takes the form

Condition 1*. All eigenvalues of Eqs.(4.4) and (4.6) are real.

It is well known /6/ that in this case the function $\delta(t)$ does not vanish on $(0, +\infty)$, Condition 2 is automatically fulfilled, and $\delta_i(t) \equiv |\delta(t)|$ $(i = 1,\ldots,m)$, while transformation A_i coincides to within sign with the identity transformation. Finally, the function $\gamma_i(t)$ introduced in Sect.2 coincides with the modulus of the function $\gamma_i(t)$ just introduced. The scalar function $(e\cdot H_i(t, z_0))$ is the difference between the solutions of differential Eq. (4.4) and of one of Eqs. (4.6), in connection with which it has, by Lemma 1, no more than $k + k_i - 1 = N_i$ zeros. Conditions 3 and 4 turn into the following.

Condition 2*. One of the eigenvalues of Eq. (4.4) is nonnegative.

Condition 3*.

$$\max_{i=1,\ldots,m}\ \sup_{r\in(0,+\infty)} \Big|\frac{\gamma_i(r)}{\delta(r)}\Big|\,\Gamma(N_1,\ldots,N_m) = \mu \in \Big(0, \frac{\sigma}{\rho}\Big);\quad \rho = \Big(\sum_{i=1}^m \rho_i^2\Big)^{1/2}$$

Let us derive a simple criterion ensuring the finiteness of number μ (and, consequently, guaranteeing, when Conditions 1* and 2* are fulfilled, the possibility of l_α-escape when the ratio δ/ρ is sufficiently large).

Lemma 4. Let $k \leqslant k_i$ $(i = 1,\ldots,m)$ and for each i let the largest eigenvalue of Eq. (4.6) not exceed the largest eigenvalue of Eq. (4.4) (and, in case they coincide, let the multiplicity of the first not exceed that of the second). Then $\mu < +\infty$.

The computation of constant μ for an analog of Pontriagin's check example

$$x_i^{\cdot\cdot} + \alpha_i x_i^{\cdot} = u_i \ (i = 1,\ldots,m),\ y^{\cdot\cdot} + \beta y^{\cdot} = v,\ x_i,\ y,\ u_i,\ v \in E,\ \dim E \geqslant 3, \quad \alpha_i > 0,\ \beta > 0$$

with constraints (4.3) yields

$$\mu = \frac{36m + 22}{\sqrt{3}} \max_{i=1,\ldots,m} \max\{1, \beta/\alpha_i\}$$

Conditions 1* and 2* are trivially verified.

5. We present an example showing that l-escape may be absent in a game with integral constraints. Consider the "child and crocodile" problem:

$$x^{\cdot\cdot} = u, \quad \int_0^{+\infty} |u(s)|^2\,ds \leqslant \rho^2, \quad y^{\cdot} = v, \quad \int_0^{+\infty} |v(s)|^2\,ds \leqslant \sigma^2, \quad (x, y, u, v \in E, \dim E \geqslant 2) \tag{5.1}$$

with a terminal set defined by the condition $x = y$. Escape is possible in problem (5.1) /5/ with an estimate of type (1.3), and $l(t) \to 0$ as $t \to +\infty$. Let us show that l-escape is impossible in game (5.1). The proof is by contradiction. Let $l_0 > 0$ and initial data $x_0 = x(0)$, $y_0 = y(0)$, $x_0^{\cdot} = x^{\cdot}(0)$ exist for which the evader can, by using his own information, ensure the inequality

$$|x(t) - y(t)| \geqslant l_0 > 0 \tag{5.2}$$

for all sufficiently large t (for definiteness, for $t \geqslant 0$). We set

$$\chi = \rho^2\sigma^{-2}, \quad \varkappa = \prod_{i=1}^{\infty}(1 - 2^{-2i-1}), \quad \varepsilon_0 = (\chi\varkappa)^{-1/2}$$

We assume that the pursuer plays the game in cycles (the i-th cycle starts at instant T_i and ends at instant $T_{i+1} = T_i + \theta_i$ which is the starting instant of the next cycle; $T_1 = 0$). At the start of the i-th cycle the pursuer determines the quantities

$$\rho_i^2 = \rho^2 - \int_0^{T_i} |u(s)|^2\,ds > 0, \quad \sigma_i^2 = \sigma^2 - \int_0^{T_i} |v(s)|^2\,ds \geqslant 0 \tag{5.3}$$

specifies the duration $\theta_i \geqslant 2\varepsilon_0$ of the cycle so as to ensure the inequality

$$|r_i| \leqslant \rho_i 2^{-i-1}\varphi(\theta_i),\ r_i = y(T_i) - x(T_i) - \theta_i x^{\cdot}(T_i),\ \varphi(s) = (s+1)\log(s+1) - s \tag{5.4}$$

(this is obviously possible for a sufficiently large θ_i), and sets his own control over the whole cycle equal to

$$u(s) = \begin{cases}(1 + s - T_i)^{-1} r_i/\varphi(\theta_i) + \dfrac{v(s - \varepsilon_0)}{T_{i+1} - s}, & s \in [T_i + \varepsilon_0, \quad T_{i+1} - \varepsilon_0] \\ (1 + s - T_i)^{-1} r_i/\varphi(\theta_i), & s \in [T_i, T_{i+1}] \setminus [T_i + \varepsilon_0, T_{i+1} - \varepsilon_0]\end{cases} \tag{5.5}$$

A direct calculation convinces us that

$$\int_{T_{i+1} - 2\varepsilon_0}^{T_{i+1}} v(s)\,ds = q_{i+1} = y(T_{i+1}) - x(T_{i+1})$$

so that by virtue of (5.2)

$$\sigma_i^2 - \sigma_{i+1}^2 \geqslant \int_{T_{i+1} - 2\varepsilon_0}^{T_{i+1}} |v(s)|^2\,ds \geqslant |q_{i+1}|^2/(2\varepsilon_0) \geqslant \frac{l_0^2}{2\varepsilon_0} \tag{5.6}$$

Let us verify that by such a method the pursuer (under the condition that the evader observes his own integral constraint, which is obligatory) has fulfilled the inequalities

$$\rho_i^2 \geqslant \sigma_i^2 \omega_i \frac{2}{\varepsilon_0^2}, \quad i = 1, 2, \ldots, \quad \omega_i = \frac{1}{\varkappa} \prod_{j=0}^{i-1} (1 - 2^{-2j-1}) \tag{5.7}$$

which leads to a contradiction between (5.3) and (5.6) (we would have $\sigma^2 \geqslant i l_0^2 (2\varepsilon_0)^{-1}$ for all i, which is absurd) and completes the proof. Indeed, for $i = 1$ inequality (5.7) is obvious. Henceforth we argue by induction. If (5.7) is fulfilled for a number i, then by definition (5.5) and inequality (5.4)

$$\rho_i^2 - \rho_{i+1}^2 \leqslant (\rho_i 2^{-i-1} + \varepsilon_0^{-1} (\sigma_i^2 - \sigma_{i+1}^2)^{1/2})^2 \leqslant \rho_i^2 2^{-2i-1} + 2\varepsilon_0^{-2}(\sigma_i^2 - \sigma_{i+1}^2)$$

in connection with which

$$1/_2\varepsilon_0^2\rho_{i+1}^2 \geqslant 1/_2\varepsilon_0^2\rho_i^2(1 - 2^{-2i-1}) - (\sigma_i^2 - \sigma_{i+1}^2) \geqslant \sigma_i^2[\omega_{i+1} - 1] + \sigma_{i+1}^2 \geqslant \sigma_{i+1}^2 \omega_{i+1}$$

as required (here we have used the nonnegativity of the expression within the brackets and the inequality $\sigma_i^2 \geqslant \sigma_{i+1}^2$).

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